

# Peakon-antipeakon interactions in the Degasperis-Procesi Equation

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## Abstract

Peakons are singular, soliton-like solutions to nonlinear wave equations whose dynamics can be studied using ordinary differential equations (ODEs). The Degasperis-Procesi equation (DP) is an important example of an integrable PDE exhibiting wave breaking in the peakon sector thus affording an interpretation of wave breaking as a mechanical collision of particles. In this paper we set up a general formalism in which to study collisions of DP peakons and apply it, as an illustration, to a detailed study of three colliding peakons. It is shown that peakons can collide only in pairs, no triple collisions are allowed and at the collision a shockpeakon is created. We also show that the initial configuration of peakon-antipeakon pairs is nontrivially correlated with the spectral properties of an accompanying non-selfadjoint boundary value problem. In particular if peakons or antipeakons are bunched up on one side relative to a remaining antipeakon or peakon then the spectrum is real and simple. Even though the spectrum is in general complex the existence of a global solution in either time direction dynamics is shown to imply the reality of the spectrum of the boundary value problem.

## 1 Introduction

The prototypical example of PDEs admitting peaked solitons is the family

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}, \quad (1.1)$$

often written as

$$m_t + m_x u + b m u_x = 0, \quad m = u - u_{xx}, \quad (1.2)$$

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which was introduced by Degasperis, Holm and Hone [1], and shown to be Hamiltonian for all values of  $b$  [2]. The most studied cases are the Camassa–Holm (CH) equation ( $b = 2$ ), and the Degasperis–Procesi (DP) equation [1, 3] ( $b = 3$ ). For  $b > 0$  these are the only values of  $b$  for which the equation is integrable, according to a variety of integrability tests [3–6]. The case  $b = 0$  is relevant for another reason; this case provides a regularization of the inviscid Burgers equation that is Hamiltonian and has classical solutions globally in time [7]. The  $b$ -family admits an appealing geometric interpretation as Euler–Arnold equations on the space of densities  $m(x)dx^b$  for the group of orientation-preserving diffeomorphisms  $\text{Diff}(S^1)$  [8].

In order to discuss peakon solutions one needs to develop the concept of weak solutions. Because of the role that the Lax pair formalism plays in the theory we will define weak solutions in such a way that the PDE in question is the compatibility condition of weak Lax pairs. This prompts the  $x$  member of the Lax pair to be viewed as an ODE with distribution coefficients while the  $t$  equation of the Lax pair is viewed as a (isospectral) deformation of the former. We subsequently need to rewrite the PDE itself as a distribution equation. To this end we observe that the formulation (1.2) suffers from the problem that the product  $mu_x$  is ill-defined already in the case of continuous, piecewise smooth (in  $x$ )  $u(x, t)$ , since the quantity  $m = u - u_{xx}$  is a measure with a non-empty singular support at the points of non-smoothness. To make matters worse that measure is in addition multiplied by the function  $u_x$  which has jump discontinuities exactly at those points. This problem can be resolved easily; one instead rewrites (1.1) as

$$(1 - \partial_x^2)u_t + (b + 1 - \partial_x^2)\partial_x\left(\frac{1}{2}u^2\right) + \partial_x\left(\frac{3-b}{2}u_x^2\right) = 0. \quad (1.3)$$

The case  $b = 3$  is of particular interest to us. Then the term  $u_x^2$  is absent from equation (1.3) and in that particular case one requires only that  $u(\cdot, t) \in L_{\text{loc}}^2(\mathbf{R})$ ; this means that the DP equation can admit solutions  $u$  that are not continuous [9–11].

*Multipeakons* are weak solutions of the form

$$u(x, t) = \sum_{i=1}^n m_i(t) e^{-|x - x_i(t)|}, \quad (1.4)$$

formed through superposition of  $n$  *peakons* (peaked solitons of the shape  $e^{-|x|}$ ). This ansatz satisfies the PDE (1.3) if and only if the positions  $(x_1, \dots, x_n)$  and momenta  $(m_1, \dots, m_n)$  of the peakons obey the following system of  $2n$  ODEs:

$$\dot{x}_k = \sum_{i=1}^n m_i e^{-|x_k - x_i|}, \quad \dot{m}_k = (b-1)m_k \sum_{i=1}^n m_i \text{sgn}(x_k - x_i) e^{-|x_k - x_i|}. \quad (1.5)$$

Here,  $\text{sgn } x$  denotes the signum function, which is  $+1$ ,  $-1$  or  $0$  depending on whether  $x$  is positive, negative or zero. In shorthand notation, with  $\langle f(x) \rangle$  denoting the average of the left and right limits,

$$\langle f(x) \rangle = \frac{1}{2}(f(x^-) + f(x^+)), \quad (1.6)$$

the ODEs can be written as

$$\dot{x}_k = u(x_k), \quad \dot{m}_k = -(b-1) m_k \langle u_x(x_k) \rangle. \quad (1.7)$$

In the CH case ( $b = 2$ ) this is a canonical Hamiltonian system generated by  $h = \frac{1}{2} \sum_{j,k=1}^n m_j m_k e^{-|x_j - x_k|}$ , for which  $x_j$ s and  $m_j$ s are canonical positions and momenta. In the DP case ( $b = 3$ ) this is a non-canonical Hamiltonian system with the Hamiltonian  $H = \sum_{j=1}^n m_j$  and a non-canonical Poisson structure given in [12].

It is important to distinguish the case of *pure peakons* (initial  $m_j(0) > 0$ ) or *pure anti-peakons* (initial  $m_j(0) < 0$ ) from a general case of *multipeakons* (no restriction on the signs of  $m_j(0)$ ). Pure peakons have peaks, pure anti-peakons have troughs while multipeakons contain both peaks and troughs.

The relevance of multipeakon solutions is that they provide a concrete model for wave breaking [13, 14]. For more information on the wave breaking phenomenon for this class of wave equations the reader is referred to [?] and [16–19]. In the CH case the distinction between pure peakons or anti-peakons and multipeakons does not result in a serious departure from the inverse spectral formulas for pure peakons. Indeed, explicit formulas for the  $n$ -peakon solution of the CH equation were derived by Beals, Sattinger and Szmigielski [20] and then extended to  $n$ -multipeakons in [14, 21] using inverse spectral methods and the theory of orthogonal polynomials. The situation for the DP equation is considerably different. The analysis of pure peakon solutions for the DP equation was accomplished by Lundmark and Szmigielski [22, 23] using inverse spectral methods and M.G. Krein’s theory of oscillatory kernels [24]. In short, in these papers, it was shown that in the DP case, when working with pure peakon or pure anti-peakon solutions, the concept of *total positivity* plays a fundamental role, for example, implying that the spectrum involved is positive and simple. For this reason going beyond the pure peakon sector of the DP will not be as straightforward as in the CH case which remains self-adjoint in the whole multipeakon sector. The DP spectral problem, by contrast, is manifestly non-selfadjoint. Yet, in addition to a general interest in modelling the wave breaking mechanism, there is another reason for studying multipeakon solutions of the DP equation: in [11] Lundmark introduced a new type of solution, a *shockpeakon solution*, which he showed for the case  $n = 2$  gives a unique entropy weak solution originating from the peakon-antipeakon solution. Therefore, multipeakon solutions can also provide us with an additional insight into the onset of shocks.

The paper is organized as follows. In Section 2, we set up the formalism for an arbitrary number  $n$  of multipeakons and elaborate on general forms of peakons with index 1 and  $n$ . We emphasize the role of the boundary value problem and its adjoint, both associated with the  $x$ -member of the Lax operator. In Section 3, we undertake a detailed study of three multipeakons. We establish the analytic character of solutions in Lemma 3.11 and describe the main properties of colliding pairs, culminating in Theorem 3.17 describing the creation of a shockpeakon at the collision. Section 4 is devoted to analysis of the spectrum of the boundary value problem, in particular we prove a signature-type Lemma

4.1 relating the signs of masses  $m_j(0)$  of colliding peakons to the real parts of eigenvalues. Finally, in Section 5 we classify different asymptotic in  $t$  behaviour of three multipeakons in terms of the sign configurations of the initial masses  $m_j(0)$ .

## 2 Lax pair and the multipeakon spectral problem

It was shown in [1] that the DP equation admits the Lax pair:

$$(\partial_x - \partial_{xxx})\Psi = zm\Psi, \quad \Psi_t = [z^{-1}(1 - \partial_x^2) + u_x - u\partial_x]\Psi. \quad (2.1)$$

In particular if  $u$  is given by the multipeakon ansatz (1.4),

$$m = 2 \sum_{i=1}^n m_i \delta_{x_i}, \quad (2.2)$$

and equations (1.7) for  $b = 3$  follow from the (distributional) compatibility of equations (2.1). The boundary conditions consistent with the asymptotic behaviour of  $\Psi$  read:

$$\Psi \sim e^x, \text{ as } x \rightarrow -\infty, \quad \Psi \text{ is bounded as } x \rightarrow +\infty. \quad (2.3)$$

To see how the implementation of these conditions leads to an isospectral problem we will trace back the most important steps in analysis of Lax pair for peakons. For more details the reader is referred to [23]. We start in the region  $x < x_1$  lying outside of the support of the discrete measure  $m$ . There, the first equation in the Lax pair can easily be solved and the boundary condition implemented by  $\Psi(x) = e^x$ . When  $x_k < x < x_{k+1}$ , we have

$$\Psi(x) = A_k(z)e^x + B_k(z) + C_k(z)e^{-x}, \quad 1 \leq k \leq n. \quad (2.4)$$

The coefficients  $A_k(z), B_k(z), C_k(z)$  are polynomials of degree  $k$  in  $z$  given by

$$\begin{pmatrix} A_k(z) \\ B_k(z) \\ C_k(z) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \sum_{p=1}^k \left[ \sum_{I \in \binom{[1, k]}{p}} \left( \prod_{i \in I} m_i \right) \left( \prod_{j=1}^{p-1} (1 - e^{x_{i_j} - x_{i_{j+1}}})^2 \right) \begin{pmatrix} 1 \\ -2e^{x_{i_p}} \\ e^{2x_{i_p}} \end{pmatrix} \right] (-z)^p, \quad (2.5)$$

where  $\binom{[1, n]}{p}$  is the set of all  $p$ -element subsets  $I = \{i_1 < \dots < i_p\}$  of  $\{1, \dots, n\}$ .

For  $x > x_n$  we will drop the subscript  $n$ , thus

$$\Psi(x) = A(z)e^x + B(z) + C(z)e^{-x}. \quad (2.6)$$

In the case of interest for us, namely  $n = 3$ , the coefficients  $A, B, C$  can be written explicitly

$$\begin{aligned} A(z) &= 1 - [m_1 + m_2 + m_3]z \\ &+ [m_1m_2(1 - e^{x_1-x_2})^2 + m_2m_3(1 - e^{x_2-x_3})^2 + m_1m_3(1 - e^{x_1-x_3})^2]z^2 \\ &- [m_1m_2m_3(1 - e^{x_1-x_2})^2(1 - e^{x_2-x_3})^2]z^3 \end{aligned} \quad (2.7a)$$

$$\begin{aligned} B(z) &= 2[m_1e^{x_1} + m_2e^{x_2} + m_3e^{x_3}]z \\ &- 2[m_1m_2(1 - e^{x_1-x_2})^2e^{x_2} + m_2m_3(1 - e^{x_2-x_3})^2e^{x_3} + m_1m_3(1 - e^{x_1-x_3})^2e^{x_3}]z^2 \\ &+ 2[m_1m_2m_3(1 - e^{x_1-x_2})^2(1 - e^{x_2-x_3})^2e^{x_3}]z^3, \end{aligned} \quad (2.7b)$$

$$\begin{aligned} C(z) &= -[m_1e^{2x_1} + m_2e^{2x_2} + m_3e^{2x_3}]z \\ &[m_1m_2(1 - e^{x_1-x_2})^2e^{2x_2} + m_2m_3(1 - e^{x_2-x_3})^2e^{2x_3} + m_1m_3(1 - e^{x_1-x_3})^2e^{2x_3}]z^2 \\ &- [m_1m_2m_3(1 - e^{x_1-x_2})^2(1 - e^{x_2-x_3})^2e^{2x_3}]z^3, \end{aligned} \quad (2.7c)$$

The  $t$  evolution of  $A, B, C$  can easily be inferred from the second equation of the Lax pair (2.1). One obtains:

$$\dot{A} = 0, \quad \dot{B} = \frac{B}{z} - 2AM_+, \quad \dot{C} = -BM_+, \quad \text{where } M_+ = \sum_{i=1}^n m_i e^{x_i}. \quad (2.8)$$

We therefore see that the asymptotic conditions (2.3) can be implemented by requiring  $A(z) = 0$ , and that this condition is preserved under the time flow (2.8), implying that the peakon equations with  $b = 3$  describe an isospectral deformation of the boundary value problem (2.3). This boundary value problem can be best studied with the help of two rational functions

**Definition 2.1.** (Weyl functions)  $\omega(z) = -\frac{B(z)}{2zA(z)}$ ,  $\zeta(z) = \frac{C(z)-B(z)}{2zA(z)}$ .

In this paper we will only use  $\omega(z)$ . For the case of pure peakons,  $m_i > 0$ , it was proved in [23] that  $\omega(z)$  is a Stieltjes transform of a measure, which subsequently played a major role in the solution of the inverse problem. If, however,  $m$  is a signed measure then  $\omega(z)$  has a more complicated structure because the spectrum is, in general, not simple or even real. Yet, nontrivial information about the dynamics of peakons can be extracted from  $\omega(z)$  without knowing its precise pole structure. To this end let us establish a simple lemma which follows trivially from equations (2.8) and the definition of  $\omega(z)$ .

**Lemma 2.2.**

$$\dot{\omega}(z) = \frac{\omega(z)}{z} + \frac{M_+}{z}. \quad (2.9)$$

From explicit formulas  $A(0) = 1, B(0) = 0$ , thus implying that 0 is a removable singular point of  $\omega(z)$ . Moreover, knowing the evolution of  $\omega(z)$  we can readily establish the time evolution of the data involved in its partial fraction decomposition.

**Theorem 2.3.** Suppose the partial fraction decomposition of  $\omega(z)$  is given:

$$\omega(z) = \sum_j \sum_{k=1}^{d_j} \frac{b_j^{(k)}(t)}{(z - \lambda_j)^k},$$

where  $d_j$  is the algebraic degeneracy of the eigenvalue  $\lambda_j$ . Then

$$b_j^{(k)}(t) = p_j^{(k)}(t) e^{\frac{t}{\lambda_j}}, \quad (2.10)$$

where  $p_j^{(k)}(t)$  is a polynomial in  $t$  of degree  $d_j - k$  or lower, and

$$\sum_j \dot{b}_j^{(1)}(t) = M_+. \quad (2.11)$$

*Proof.* Combining the partial fraction decomposition with (2.9) one gets

$$\frac{\omega(z)}{z} = -\frac{M_+}{z} + \sum_j \sum_{k=1}^{d_j} \frac{\dot{b}_j^{(k)}(t)}{(z - \lambda_j)^k}. \quad (2.12)$$

By Residue Theorem, we have  $0 = \text{Res}\left(\frac{\omega(z)}{z}, \infty\right) + \text{Res}\left(\frac{\omega(z)}{z}, 0\right) + \sum_j \text{Res}\left(\frac{\omega(z)}{z}, \lambda_j\right)$  where

$$\text{Res}\left(\frac{\omega(z)}{z}, \infty\right) = 0, \quad \text{Res}\left(\frac{\omega(z)}{z}, 0\right) = -M_+, \quad \text{Res}\left(\frac{\omega(z)}{z}, \lambda_j\right) = \dot{b}_j^{(1)}(t),$$

which proves (2.11).

By the formulas for the coefficients in the Laurant series of equation (2.12) we obtain

$$\dot{b}_j^{(k)}(t) = \sum_{s=k}^{d_j} \frac{1}{(s-k)!} \frac{d^{s-k}}{dz^{s-k}} \left( \frac{b_j^{(s)}(t)}{z} \right) \Big|_{z=\lambda_j} = \sum_{s=k}^{d_j} \frac{(-1)^{s-k}}{\lambda_j^{s-k+1}} b_j^{(s)}(t), \quad 1 \leq k \leq d_j.$$

In particular, we have  $\dot{b}_j^{(d_j)}(t) = \frac{b_j^{(d_j)}(t)}{\lambda_j}$ , hence  $b_j^{(d_j)}(t) = b_j^{(d_j)}(0) e^{\frac{t}{\lambda_j}}$ . Proceeding by induction we obtain

$$\dot{b}_j^{(k)}(t) = \sum_{s=k}^{d_j} \frac{(-1)^{s-k}}{\lambda_j^{s-k+1}} b_j^{(s)}(t) = \frac{b_j^{(k)}(t)}{\lambda_j} + e^{\frac{t}{\lambda_j}} \sum_{s=k+1}^{d_j} \frac{(-1)^{s-k}}{\lambda_j^{s-k+1}} p_j^{(s)}(t) \stackrel{\text{def}}{=} \frac{b_j^{(k)}(t)}{\lambda_j} + e^{\frac{t}{\lambda_j}} \tilde{p}^{(k+1)}(t),$$

therefore  $b_j^{(k)}(t) = e^{\frac{t}{\lambda_j}} \left( b_j^{(k)}(0) + \int_0^t \tilde{p}^{(k+1)}(\tau) d\tau \right) \stackrel{\text{def}}{=} e^{\frac{t}{\lambda_j}} p_j^{(k)}(t)$ . Finally, since  $\tilde{p}^{(k+1)}(t)$  is a polynomial of degree  $d_j - k - 1$  or lower,  $p_j^{(k)}(t)$  is a polynomial of degree  $d_j - k$  or lower, which leads to (2.10).  $\square$

**Lemma 2.4.** *Let  $x_n$  be the position of the  $n$ -th mass. Then*

$$e^{x_n} = \sum_j b_j^{(1)}. \quad (2.13)$$

*Proof.* By Residue Theorem  $0 = \text{Res}(\omega(z), \infty) + \sum_j \text{Res}(\omega(z), \lambda_j)$ . Thus  $\sum_j b_j^{(1)} = -\text{Res}(\omega(z), \infty)$ . With the help of explicit formulas (2.5) and the definition of  $\omega(z)$  we obtain

$$\text{Res}(\omega(z), \infty) = -\lim_{z \rightarrow \infty} z\omega(z) = \lim_{z \rightarrow \infty} \frac{B(z)}{2A(z)} = -e^{x_n},$$

which proves the conclusion.  $\square$

**Corollary 2.5.** *The  $n$ th mass cannot escape to  $+\infty$  in finite real time.*

*Proof.* Indeed, from Theorem 2.3 and the lemma above we see that  $e^{x_n}$  has at most an exponential growth, hence it is bounded for finite real time.  $\square$

To deal with the behaviour of  $x_1$  we will use a slightly modified spectral problem which, in principle, amounts to “sweeping” the masses in the opposite direction. To this end we consider the adjoint Lax pair:

$$(\partial_x - \partial_{xx})\tilde{\Psi} = -zm\tilde{\Psi}, \quad \tilde{\Psi}_t = [-z^{-1}(1 - \partial_x^2) + u_x - u\partial_x]\tilde{\Psi}. \quad (2.14)$$

**Remark 2.6.** The only difference between equations (2.1) and (2.14) is the sign of  $z$  which has no effect on the compatibility conditions; hence the adjoint Lax pair gives the same compatibility condition — the DP equation.

We choose a different set of asymptotic conditions, namely

$$\tilde{\Psi} \sim e^{-x}, \text{ as } x \rightarrow +\infty, \quad \tilde{\Psi} \text{ is bounded as } x \rightarrow -\infty. \quad (2.15)$$

For  $x < x_1$

$$\tilde{\Psi}(x) = \tilde{A}(z)e^{-x} + \tilde{B}(z) + \tilde{C}(z)e^x. \quad (2.16)$$

Hence the adjoint spectral problem is given by  $\tilde{A}(z) = 0$ . Likewise, one readily checks that the time flow given by the second equation in (2.14) yields:

$$\dot{\tilde{A}} = 0, \quad \dot{\tilde{B}} = -\frac{\tilde{B}}{z} + 2\tilde{A}M_-, \quad \dot{\tilde{C}} = \tilde{B}M_-, \text{ where } M_- = \sum_i m_i e^{-x_i}. \quad (2.17)$$

We conclude that the adjoint boundary value problem (2.15) is also isospectral under the DP flow. In fact, the spectral problems (2.3) and (2.15) have identical spectra. To demonstrate that we establish first an elementary lemma.

**Lemma 2.7.** *If  $\Psi(x)$  is the solution to the  $x$ -equation in the boundary value problem (2.1) with  $m(x) = \sum_{i=1}^n m_i \delta_{x_i}$ , then  $\Psi(-x)$  is the solution to the  $x$ -equation in (2.14) with  $\tilde{m}(x) = \sum_{i=1}^n m_i \delta_{-x_i}$  and boundary conditions (2.15).*

*Proof.* Since  $\Psi(x)$  is the solution to (2.1), the boundary conditions

$$\Psi(-x) \rightarrow e^{-x}, \text{ as } x \rightarrow +\infty, \quad \Psi(-x) \text{ is bounded as } x \rightarrow -\infty$$

hold. Moreover, we have

$$(\partial_x - \partial_{xxx})\Psi(-x) = -(\Psi_x(-x) - \Psi_{xxx}(-x)) = -zm(-x)\Psi(-x).$$

Notice that  $\delta_{x_i}(-x) = \delta_{-x_i}(x)$ , hence  $m(-x) = \tilde{m}(x)$  and the conclusion holds.  $\square$

Denote  $\underline{m} = (m_1, \dots, m_n)$ ,  $\underline{x} = (x_1, \dots, x_n)$  for short, and set  $\underline{m}^\tau, \underline{x}^\tau$  to be the vector with the reversed order of its entries, that is  $\underline{m}^\tau = (m_n, \dots, m_1)$  etc. Employing the same convention as in equation (2.4), but this time for  $\Psi$ , we obtain the following analogue of equation (2.5).

**Theorem 2.8.** *Let  $\tilde{A}_k(z) = \tilde{A}_k(z; \underline{m}, \underline{x})$ ,  $1 \leq k \leq n$ . Then*

$$\begin{pmatrix} \tilde{A}_k(z) \\ \tilde{B}_k(z) \\ \tilde{C}_k(z) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \sum_{p=1}^k \left[ \sum_{I \in \binom{[1, k]}{p}} \left( \prod_{i \in I} m_i \right) \left( \prod_{j=1}^{p-1} (1 - e^{x_{i_j} - x_{i_{j+1}}})^2 \right) \begin{pmatrix} 1 \\ -2e^{-x_{i_1}} \\ e^{-2x_{i_1}} \end{pmatrix} \right] (-z)^p,$$

where  $\binom{[1, n]}{p}$  is the set of all  $p$ -element subsets  $I = \{i_1 < \dots < i_p\}$  of  $\{1, \dots, n\}$ . In particular, when  $k = n$ ,  $A(z) = \tilde{A}(z)$ .

*Proof.* By lemma 2.7  $\tilde{\Psi}(x)$  in the asymptotic region  $x \rightarrow -\infty$  can be expressed as

$$\begin{aligned} \tilde{\Psi}(x; \underline{m}, \underline{x}) &= \tilde{A}(z; \underline{m}, \underline{x})e^{-x} + \tilde{B}(z; \underline{m}, \underline{x}) + \tilde{C}(z; \underline{m}, \underline{x})e^x \\ &= \Psi(-x; \underline{m}^\tau, -\underline{x}^\tau) = A(z; \underline{m}^\tau, -\underline{x}^\tau)e^{-x} + B(z; \underline{m}^\tau, -\underline{x}^\tau) + C(z; \underline{m}^\tau, -\underline{x}^\tau)e^x, \end{aligned}$$

which leads to  $\tilde{A}(z; \underline{m}, \underline{x}) = A(z; \underline{m}^\tau, -\underline{x}^\tau)$ ,  $\tilde{B}(z; \underline{m}, \underline{x}) = B(z; \underline{m}^\tau, -\underline{x}^\tau)$ , and  $\tilde{C}(z; \underline{m}, \underline{x}) = C(z; \underline{m}^\tau, -\underline{x}^\tau)$ . The conclusion then directly follows from the formulas (2.5) along with an elementary observation that the permutation  $\tau$  is a bijection on the ordered  $p$ -tuples, which for any fixed  $p$ -tuple maps the last element  $e^{x_{i_p}}$  in the original sum into the first element  $e^{-x_{n+1-i_p}}$  of the new  $p$ -tuple. After a simple change of index the main claim is proven. As to  $A_n(z)$ , which corresponds to the first line in the formula, we observe that this polynomial is invariant under the transformation  $\underline{m} \mapsto \underline{m}^\tau, \underline{x} \mapsto -\underline{x}^\tau$ .  $\square$

We can thus define the adjoint Weyl function  $\tilde{\omega}(z) = -\frac{\tilde{B}(z)}{2z\tilde{A}(z)}$ , and use equations (2.17) to determine the time flow of  $\tilde{\omega}$ . An easy computation gives:

**Lemma 2.9.**

$$\dot{\tilde{\omega}}(z) = -\frac{\tilde{\omega}(z)}{z} - \frac{M_-}{z}. \quad (2.18)$$

Consequently, we obtain an analogue of Theorem 2.3.



**Theorem 2.10.** *Suppose the partial fraction decomposition of  $\tilde{\omega}(z)$  is given:*

$$\tilde{\omega}(z) = \sum_j \sum_{k=1}^{d_j} \frac{\tilde{b}_j^{(k)}}{(z - \lambda_j)^k}$$

where  $d_j$  is the algebraic degeneracy of the eigenvalue  $\lambda_j$ . Then

$$\tilde{b}_j^{(k)} = \tilde{p}_j^{(k)}(t) e^{-\frac{t}{\lambda_j}} \quad (2.19)$$

where  $\tilde{p}_j^{(k)}(t)$  is a polynomial in  $t$  of degree  $d_j - k$  or lower, and

$$\sum_j \dot{\tilde{b}}_j^{(1)} = -M_-. \quad (2.20)$$

With the help of Theorem 2.8 it is now straightforward to establish a counterpart of Lemma 2.4.

**Lemma 2.11.** *Let  $x_1$  be the position of the first mass. Then*

$$e^{-x_1} = \sum_j \tilde{b}_j^{(1)}(t). \quad (2.21)$$

Likewise, an analogue of Corollary 2.5 is immediate.

**Corollary 2.12.** *The first mass cannot escape to  $-\infty$  in finite real time.*

**Example 2.13.** Case  $n = 3$ . In this case we are only dealing with simple and quadratic roots, since the triple roots cannot occur as will be proved in Section 4. The formulas for  $e^{x_3(t)}$  and  $e^{-x_1(t)}$  read:

$$e^{x_3(t)} = \begin{cases} b_1^{(1)}(0)e^{\frac{t}{\lambda_1}} + b_2^{(1)}(0)e^{\frac{t}{\lambda_2}} + b_3^{(1)}(0)e^{\frac{t}{\lambda_3}}, & \text{simple roots,} \\ b_1^{(1)}(0)e^{\frac{t}{\lambda_1}} + (b_2^{(1)}(0) - \frac{b_2^{(2)}(0)t}{\lambda_2^2})e^{\frac{t}{\lambda_2}}, & \text{quadratic roots.} \end{cases}$$

$$e^{-x_1(t)} = \begin{cases} \tilde{b}_1^{(1)}(0)e^{-\frac{t}{\lambda_1}} + \tilde{b}_2^{(1)}(0)e^{-\frac{t}{\lambda_2}} + \tilde{b}_3^{(1)}(0)e^{-\frac{t}{\lambda_3}}, & \text{simple roots,} \\ \tilde{b}_1^{(1)}(0)e^{-\frac{t}{\lambda_1}} + (\tilde{b}_2^{(1)}(0) + \frac{\tilde{b}_2^{(2)}(0)t}{\lambda_2^2})e^{-\frac{t}{\lambda_2}}, & \text{quadratic roots.} \end{cases}$$

The spectral problem and its adjoint are clearly related and we turn now to establishing a relation between them. To this end we study the coefficients occurring in the eigenfunctions of the spectral problem (2.6) and (2.16).

**Lemma 2.14.**

$$2A(z)C(-z) + 2A(-z)C(z) - B(z)B(-z) = 0,$$

$$2\tilde{A}(z)\tilde{C}(-z) + 2\tilde{A}(-z)\tilde{C}(z) - \tilde{B}(z)\tilde{B}(-z) = 0.$$

*Proof.* It suffices to write  $(D - D^3)\psi(x; z) = zm\psi(x; z), (D - D^3)\psi(x; \lambda) = \lambda m\psi(x; \lambda)$  and obtain from it the identity:

$$D(\psi(x; z)\psi(x; \lambda) - B(\psi(x; z), \psi(x; \lambda))) = (\lambda + z)m\psi(x; z)\psi(x; \lambda),$$

where  $B(f, g) = f''g - f'g' + fg''$ . Finally, if one sets  $z + \lambda = 0$  and evaluates the above expression at  $x \rightarrow -\infty$  and  $x \rightarrow \infty$  one obtains the first claim. The proof of the second identity is analogous.  $\square$

We will briefly study the symmetry responsible for the connection between the boundary value problem (2.3) and its adjoint (2.15). To this end we recall the *transition* matrix  $S(z)$  introduced in [23]

$$S(z) = S_n(z)S_{n-1}(z) \cdots S_1(z), \text{ where } \begin{bmatrix} A_k \\ B_k \\ C_k \end{bmatrix} = S_k(z) \begin{bmatrix} A_{k-1} \\ B_{k-1} \\ C_{k-1} \end{bmatrix},$$

where  $A_0 = 1, B_0 = C_0 = 0$ . An explicit form of  $S_k(z)$  is easy to compute:

$$S_k(z) = I - zm_k \begin{bmatrix} e^{-x_k} \\ -2 \\ e^{x_k} \end{bmatrix} \begin{bmatrix} e^{x_k} & 1 & e^{-x_k} \end{bmatrix}. \quad (2.22)$$

Define now

**Definition 2.15.**  $J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$

We can define the loop group of continuous maps  $G : \mathbf{R} \rightarrow \text{SL}(3, \mathbf{R})$ ; clearly  $S(z) \in G$ . Moreover, if we introduce involution:  $\tau : G \rightarrow G, g(z) \rightarrow J(g^{-1}(-z))^T J^{-1}$ , then  $S_k(z) \in G_\tau = \{g = \tau(g)\}$ , a subgroup fixed by  $\tau$ . Hence

**Lemma 2.16.**  $S(z) \in G_\tau$ .

Let us denote the canonical basis  $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  by the shorthand notation  $|1\rangle, |2\rangle, |3\rangle$ . Then equations (2.6), (2.16) can be written

$$\begin{bmatrix} A(z) \\ B(z) \\ C(z) \end{bmatrix} = S(z)|1\rangle \quad \begin{bmatrix} \tilde{C}(z) \\ \tilde{B}(z) \\ \tilde{A}(z) \end{bmatrix} = \tilde{S}(z)|3\rangle, \quad (2.23)$$

where

$$\tilde{S}(z) = S_1^{-1}(-z) \cdots S_n^{-1}(-z) = S(-z)^{-1}. \quad (2.24)$$

Moreover, since  $S_k(z) \in G_\tau$ ,

$$\tilde{S}(z) = JS^T(z)J^{-1}. \quad (2.25)$$

This is a fundamental relation which allows one to relate the spectral data for the boundary problem and its adjoint.

**Theorem 2.17.**

$$1. \tilde{A}(z) = S(z)_{11}, \quad \tilde{B}(z) = -2S(z)_{12}, \quad \tilde{C}(z) = S(z)_{13}.$$

2. Suppose  $\lambda_i$  is a root of  $A(z) = 0$  then

$$B(-\lambda_i) = C(\lambda_i)\tilde{B}(\lambda_i). \quad (2.26)$$

3. Suppose  $B(-\lambda_i) \neq 0$  then

$$2A(-\lambda_i) = B(\lambda_i)\tilde{B}(\lambda_i). \quad (2.27)$$

*Proof.* By definition  $\tilde{A}(z) = \langle 3|JS^T(z)J^{-1}|3\rangle = \langle 1|S^T(z)|1\rangle = S(z)_{11}$ . Likewise,  $\tilde{B}(z) = \langle 2|JS^T(z)J^{-1}|3\rangle = -2\langle 2|S^T(z)|1\rangle = -2S(z)_{12}$  and  $\tilde{C}(z) = \langle 1|JS^T(z)J^{-1}|3\rangle = S(z)_{13}$ . The second item is proved by making use of the involution  $\tau$ . On one hand  $B(z) = \langle 2|S(z)|1\rangle$ , on the other, since  $S(z) \in G_\tau$ ,  $B(z) = -2\langle 3|S^{-1}(z)|2\rangle = 2 \begin{vmatrix} S(-z)_{11} & S(-z)_{12} \\ S(-z)_{31} & S(-z)_{32} \end{vmatrix}$ . Finally, since  $S(z)_{11} = A(z)$ , evaluating the determinant at the (minus) root  $\lambda_i$  of  $A(z)$  we obtain  $B(-\lambda_i) = -2S(\lambda_i)_{12}S(\lambda_i)_{31} = \tilde{B}(\lambda_i)C(\lambda_i)$ , in view of the statement from item (1). Finally, to prove item (3), we set  $z = \lambda_i$  in the statement of Lemma 2.14 to get  $2A(-\lambda_i)C(\lambda_i) = B(\lambda_i)B(-\lambda_i)$ . Upon multiplying equation (2.26) by  $2A(-\lambda_i)$  and eliminating the term involving  $2A(-\lambda_i)C(\lambda_i)$  we obtain  $2A(-\lambda_i)B(-\lambda_i) = B(-\lambda_i)B(\lambda_i)\tilde{B}(\lambda_i)$ , resulting in equation (2.27).  $\square$

Consider now the Weyl function  $\omega(z)$  and its adjoint  $\tilde{\omega}(z)$  in the case of simple spectrum, i.e.

$$\omega(z) = -\frac{B(z)}{2zA(z)} = \sum_{i=1}^n \frac{b_i}{z - \lambda_i} \text{ and } \tilde{\omega}(z) = -\frac{\tilde{B}(z)}{2z\tilde{A}(z)} = \sum_{i=1}^n \frac{\tilde{b}_i}{z - \lambda_i}.$$

**Theorem 2.18.** Suppose the spectral problem  $A(z) = 0$  has only simple roots  $\lambda_i$  and there are no anti-resonances ( $\lambda_i + \lambda_j \neq 0$ ). Then

$$b_i\tilde{b}_i = \prod_{j \neq i} \frac{1 + \frac{\lambda_i}{\lambda_j}}{(1 - \frac{\lambda_i}{\lambda_j})^2}. \quad (2.28)$$

*Proof.* Under the assumption of simple spectrum:

$$b_i\tilde{b}_i = \frac{B(\lambda_i)\tilde{B}(\lambda_i)}{4\lambda_i^2(A'(\lambda_i))^2}$$

which simplifies, after using equation (2.27), to

$$b_i\tilde{b}_i = \frac{A(-\lambda_i)}{2\lambda_i^2(A'(\lambda_i))^2} = \frac{\prod_{j=1}^n (1 + \frac{\lambda_i}{\lambda_j})}{2 \prod_{j \neq i} (1 - \frac{\lambda_i}{\lambda_j})^2},$$

which implies the claim if one observes that the term with  $j = i$  appearing in the numerator contributes the factor of 2 canceling the one from the denominator.  $\square$

**Remark 2.19.** This beautiful identity generalizes the one known from the ordinary string problem [25] which in our notation reads:

$$b_i \tilde{b}_i = \prod_{j \neq i} \left(1 - \frac{\lambda_i}{\lambda_j}\right)^{-2}.$$

**Remark 2.20.** The presence of anti-resonances ( $\lambda_i + \lambda_j = 0$ ) is characteristic of the DP equation as can be seen, for example, from explicit solutions.

### 3 Three multipeakons

In this section we apply the methods developed in Section 2 to study three multipeakons, with emphasis on the behaviour of solutions at the time of blow-up. As before we use the multipeakon ansatz (1.4)

$$u(x, t) = \sum_{i=1}^3 m_i(t) e^{-|x-x_i(t)|} \quad (3.1)$$

where  $x_1(0) < x_2(0) < x_3(0)$ , and we no longer assume that  $m_i(t)$  are all positive. In spite of that we will refer to  $m_j$ s as *masses* to emphasize their roles in the spectral problem. We will need a bit of terminology regarding the phenomenon of breaking. Since we will be analyzing a system of ODEs obtained from a restriction of equation (1.7) we will say that at some time  $t_0$  a *collision* occurred if for some  $i \neq j$ ,  $x_i(t_0) = x_j(t_0)$ . In the case of the CH equation the presence of a collision is tantamount to a wave breaking ([14]) but the solution can be continued with the preservation of the Sobolev  $H^1(\mathbf{R})$  norm beyond the collision time. This is not the case for the DP equation as was anticipated by Lundmark in [11] for the case of the peakon-antipeakon pair. We confirm his assertion that the shockpeakons are created by proving that  $m = u - u_{xx}$  tends to the shockpeakon data in the distribution topology at the collision time (see Theorem 3.17).

We start by setting  $b = 3$  and  $n = 3$  in the multipeakon equation (1.7), which leads to the following ODEs in the sector  $X = \{\mathbf{x} \in \mathbf{R}^3 \mid x_1 < x_2 < x_3\}$ :

$$\dot{x}_1 = m_1 + m_2 e^{x_1-x_2} + m_3 e^{x_1-x_3}, \quad (3.2a)$$

$$\dot{x}_2 = m_1 e^{x_1-x_2} + m_2 + m_3 e^{x_2-x_3}, \quad (3.2b)$$

$$\dot{x}_3 = m_1 e^{x_1-x_3} + m_2 e^{x_2-x_3} + m_3, \quad (3.2c)$$

$$\dot{m}_1 = 2m_1(-m_2 e^{x_1-x_2} - m_3 e^{x_1-x_3}), \quad (3.2d)$$

$$\dot{m}_2 = 2m_2(m_1 e^{x_1-x_2} - m_3 e^{x_2-x_3}), \quad (3.2e)$$

$$\dot{m}_3 = 2m_3(m_1 e^{x_1-x_3} + m_2 e^{x_2-x_3}). \quad (3.2f)$$

This system of equations has the following obvious symmetry.

**Lemma 3.1.** *Suppose  $\{x_1(t), x_2(t), x_3(t), m_1(t), m_2(t), m_3(t)\}$  is a solution of equations (3.2) at time  $t$  with the initial condition  $\{x_1(0), x_2(0), x_3(0), m_1(0), m_2(0), m_3(0)\}$ .*

Then  $\{x_1(t), x_2(t), x_3(t), -m_1(t), -m_2(t), -m_3(t)\}$  is the solution at time  $-t$  with the initial condition  $\{x_1(0), x_2(0), x_3(0), -m_1(0), -m_2(0), -m_3(0)\}$ .

**Remark 3.2.** In short, the lemma above means that  $t \mapsto -t, m_i \mapsto -m_i$  is a symmetry of equations (3.2) which preserves the sector  $X$ .

Another very useful property of equations (3.2) is the existence of three constants of motion. Indeed, we recall that the polynomial  $A(z)$  introduced in (2.7) is time invariant. Writing

$$A(z) = 1 - M_1 z + M_2 z^2 - M_3 z^3 \quad (3.3)$$

we obtain the following lemma.

**Lemma 3.3.**  $M_1, M_2, M_3$ , given by:

$$\begin{aligned} M_1 &= m_1 + m_2 + m_3, \\ M_2 &= m_1 m_2 (1 - e^{x_1 - x_2})^2 + m_2 m_3 (1 - e^{x_2 - x_3})^2 + m_3 m_1 (1 - e^{x_1 - x_3})^2, \\ M_3 &= m_1 m_2 m_3 (1 - e^{x_1 - x_2})^2 (1 - e^{x_2 - x_3})^2, \end{aligned}$$

are constants of motion of the system of equations (3.2).

These constants will be one of our basic tools for studying collisions. We observe that, geometrically speaking, a collision occurs if the solution approaches the boundary of  $X$  in finite time. This is the only singular behaviour of the system (3.2) happening in the coordinate space since Corollaries 2.5 and 2.12 exclude an escape scenario in finite real time. However, the shape of the constants of motion shows that at a collision at least two masses diverge, which will be proved in Corollary 3.16.

We begin now our study of the dynamics of three multipeakons in a vicinity of the collision by first concentrating on the particles with labels 1 and 3. Lemma 2.4, in particular equation (2.13), gives us explicit form of  $x_3$ :

$$e^{x_3(t)} = \sum_j b_j^{(1)}(t). \quad (3.4)$$

Thus we obtain:

**Lemma 3.4.** Let  $T_1, T_2$ , be the largest negative, respectively the smallest positive root of  $\sum_j b_j^{(1)}(t) = 0$  (if  $T_1$  or  $T_2$  does not exist we set  $T_1 = -\infty, T_2 = +\infty$  respectively). Then  $e^{x_3(t)}$  is real analytic for  $T_1 < t < T_2$ . Moreover, if either  $T_1$  or  $T_2$  are finite then there must be a collision at some prior time  $T_1 < t_c < T_2$ .

**Remark 3.5.** For positive  $t$  “prior” has the usual meaning (positive orientation). For negative  $t$  the orientation is from 0 to  $-\infty$ .

*Proof.* The last statement follows from Corollary 2.12, since finite  $T_1$  or  $T_2$  means that  $x_3$  escaped to  $-\infty$  which cannot happen in finite time unless there is a collision at an earlier time.  $\square$

**Remark 3.6.** Since  $b_j^{(1)}(t)$  is an exponential function of  $t$ , the right hand side of (3.4) is well define for any real  $t$ . However the left hand side of (3.4) only make sense when  $t$  lies in the existence interval of the ODE system (3.2).

Furthermore, combining equations for  $\dot{x}_3$  and  $\dot{m}_3$  (see equations (3.2)) yields:

$$\dot{m}_3 = 2m_3(\dot{x}_3 - m_3). \quad (3.5)$$

We remark that this is a Bernoulli type equation which can be easily solved once  $x_3(t)$  is known.

**Lemma 3.7.** *Suppose  $x_3(t)$  is known. Then*

$$\frac{1}{m_3(t)} = e^{-2x_3(t)} \left[ \frac{e^{2x_3(0)}}{m_3(0)} + 2 \int_0^t e^{2x_3(\tau)} d\tau \right] \quad (3.6)$$

An analogous argument works for  $x_1$ . Indeed, by equation (2.21), we have

$$e^{-x_1(t)} = \sum_j \tilde{b}_j^{(1)}.$$

This prompts an analogous statement to Lemma 3.4

**Lemma 3.8.** *Let  $\tilde{T}_1, \tilde{T}_2$ , be the largest negative, respectively the smallest positive, root of  $\sum_j \tilde{b}_j^{(1)}(t) = 0$ . Then  $e^{-x_1(t)}$  is real analytic for  $\tilde{T}_1 < t < \tilde{T}_2$ . Moreover, if either  $\tilde{T}_1$  or  $\tilde{T}_2$  are finite then there must be a collision at some prior time  $\tilde{T}_1 < t_c < \tilde{T}_2$ .*

We see that we can now narrow down the time of a collision. Let us denote by  $A = (T_1, T_2) \cap (\tilde{T}_1, \tilde{T}_2)$ . We summarize analytic properties of  $e^{x_3}$  and  $e^{-x_1}$ .

**Lemma 3.9.** *The functions  $e^{x_3(t)}$  and  $e^{-x_1(t)}$  are real analytic on  $A$ . A collision can only occur at a time  $t_c$  if  $t_c \in A$ . In particular, both functions are analytic at the time of collision.*

Once again, if we know  $x_1$  then we can determine  $m_1$ .

Indeed, equations (3.2) imply another Bernoulli equation:

$$\dot{m}_1 = -2m_1(\dot{x}_1 - m_1), \quad (3.7)$$

whose solution reads

**Lemma 3.10.**

$$\frac{1}{m_1(t)} = e^{2x_1(t)} \left[ \frac{e^{-2x_1(0)}}{m_1(0)} - 2 \int_0^t e^{-2x_1(\tau)} d\tau \right]. \quad (3.8)$$

We can now summarize analytic properties of  $m_1, m_2$  and  $m_3$ .

**Lemma 3.11.**

- (1)  $\frac{1}{m_1(t)}$  and  $\frac{1}{m_3(t)}$  are real analytic on  $A$ .
- (2)  $m_2(t)$  and  $e^{x_2(t)}$  are real meromorphic functions on  $A$ .
- (3) A collision occurs iff there exists  $t_c \in A$  such that either  $\frac{1}{m_1(t_c)} = 0$  or  $\frac{1}{m_3(t_c)} = 0$ .
- (4) Suppose  $\frac{1}{m_1(t_c)} = 0$ . Then in a neighborhood of  $t_c$

$$\frac{1}{m_1(t)} = -2(t - t_c) + \mathcal{O}((t - t_c)^2).$$

- (5) Suppose  $\frac{1}{m_3(t_c)} = 0$ . Then in a neighborhood of  $t_c$

$$\frac{1}{m_3(t)} = 2(t - t_c) + \mathcal{O}((t - t_c)^2).$$

*Proof.* The analytic properties of  $\frac{1}{m_1}$  and  $\frac{1}{m_3}$  are directly derived from the analytic properties of  $e^{x_1(t)}$  and  $e^{x_3(t)}$  and Lemmas 3.10, 3.7 respectively.

To prove (2) we note that using  $M_1$  we can write  $m_2 = M_1 - (m_1 + m_3)$ . Likewise  $e^{x_2}$  can be easily computed from  $M_+ = \sum_j m_j e^{x_j} = \frac{d}{dt} e^{x_3}$  by algebraic operations on analytic functions.

To see (3), since  $M_3$  is a constant of motion, we observe that a collision occurs when at least two of the masses diverge, hence either  $m_1$  or  $m_3$  have to diverge at a collision.

To prove (4) and (5) we only need to calculate the derivatives of  $\frac{1}{m_1(t)}$  and  $\frac{1}{m_3(t)}$  at  $t_c$ . Since  $x_1, x_3$  are analytic at  $t_c$ , direct computation from (3.5) and (3.7) shows that

$$\begin{aligned} \left. \frac{d}{dt} \frac{1}{m_3(t)} \right|_{t=t_c} &= \left( 2 - \frac{2\dot{x}_3}{m_3} \right) \Big|_{t=t_c} = 2, \\ \left. \frac{d}{dt} \frac{1}{m_1(t)} \right|_{t=t_c} &= \left( \frac{2\dot{x}_1}{m_1} - 2 \right) \Big|_{t=t_c} = -2. \end{aligned}$$

Therefore (3) and (4) hold. □

Now we can obtain the behaviour of masses before a collision. As a general comment, we observe that for any initial data in  $X$  and arbitrary  $m_1, m_2, m_3$  the solution is unique. From this point onwards we assume  $m_i(0) \neq 0$ . Thus  $M_3$  is nonzero.

**Lemma 3.12.** *None of the masses  $m_i$  can become zero before a collision.*

*Proof.* In order to derive a contradiction, we suppose that one of the masses becomes zero at  $t_0$ . Since all three constants of motion given by Lemma 3.3 are symmetric with respect to permutations of masses, we can assume, without a loss of generality, that  $m_1(t_0) = 0$ . Since  $M_3 \neq 0$ ,  $m_2 m_3$  diverges at  $t_0$ . Then  $\frac{M_2}{m_2 m_3}$  converges to zero, while the corresponding right hand side converges to  $(1 - e^{x_2 - x_3})^2 \neq 0$ , thus a contradiction.  $\square$

**Corollary 3.13.** *The masses  $m_i$  cannot change their signs before a collision.*

**Corollary 3.14.** *None of the masses  $m_i$  will diverge to  $\pm\infty$  before a collision.*

*Proof.* Since none of  $m_i$  can become zero by Lemma 3.12,  $M_3$  is nonzero,  $0 < \frac{M_3}{m_1 m_2 m_3} < \infty$  and  $\frac{M_3}{m_1 m_2 m_3} = (1 - e^{x_1 - x_2})^2 (1 - e^{x_2 - x_3})^2$  is continuous before a collision, hence the claim follows.  $\square$

Combined with the analytic property, there are two corollaries worth mentioning.

**Corollary 3.15** (Absence of triple collisions). *There are no triple collisions, that is, there is no time at which  $x_1 = x_2 = x_3$ .*

*Proof.* Suppose  $x_1(t_c) = x_2(t_c) = x_3(t_c)$ . Then the leading contribution to  $M_3$  coming from the term  $(1 - e^{x_1 - x_2})^2 (1 - e^{x_2 - x_3})^2$  is  $\mathcal{O}((t - t_c)^4)$  which forces  $m_1 m_2 m_3$  to behave like  $\mathcal{O}(\frac{1}{(t - t_c)^4})$ . If only  $m_1, m_2$  diverge then  $m_1$  diverges as  $\frac{1}{2(t_c - t)}$  by Lemma 3.11 and  $m_2$  would have to diverge as  $\mathcal{O}(\frac{1}{(t - t_c)^3})$  violating conservation of  $M_1$ . Similar argument excludes divergence of  $m_2, m_3$ . The last case is that  $m_1, m_2, m_3$  diverge, but then in view of Lemma 3.11  $m_2$  would have to diverge as  $\mathcal{O}(\frac{1}{(t - t_c)^2})$ , again violating conservation of  $M_1$ .  $\square$

**Corollary 3.16.** *At the point of a collision masses diverge in pairs and the only admissible pairs are  $\{m_1, m_2\}$  and  $\{m_2, m_3\}$ .*

*Proof.* In view of the behaviour of  $m_1$  and  $m_3$  at the collision,  $m_2$  must be regular to preserve  $M_1$  if  $m_1$  and  $m_3$  diverge. Thus  $m_1, m_2, m_3$  cannot all diverge. To eliminate the  $m_1, m_3$  pair we consider  $0 = \lim_{t \rightarrow t_c} \frac{M_2}{m_1 m_3} = (1 - e^{x_1(t_c) - x_3(t_c)})^2 \neq 0$  by the absence of triple collisions, hence a contradiction.  $\square$

**Theorem 3.17** (Shockpeakon creation). *If  $m_j$  collides with  $m_{j+1}$  at  $t_c > 0$ , then*

$$\begin{aligned} & \lim_{t \rightarrow t_c^-} (m_j(t) \delta(x - x_j(t)) + m_{j+1}(t) \delta(x - x_{j+1}(t))) \\ &= \left( \lim_{t \rightarrow t_c^-} (m_j + m_{j+1}) \right) \delta(x - x(t_c)) + \frac{1}{2} \left( \lim_{t \rightarrow t_c^-} (u(x_j(t), t) - u(x_{j+1}(t), t)) \right) \delta'(x - x(t_c)), \end{aligned}$$

where the limit is in the sense of  $\mathcal{D}'(\mathbb{R})$ .



*Proof.* For arbitrary  $\varphi(x) \in \mathcal{D}(\mathbb{R})$ ,

$$\langle m_j(t)\delta(x-x_j(t))+m_{j+i}(t)\delta(x-x_{j+1}(t)), \varphi(x) \rangle = m_j(t)\varphi(x_j(t))+m_{j+i}(t)\varphi(x_{j+1}(t)).$$

Whenever  $j = 1$  or  $2$ , we can always write

$$m_j = -\frac{1}{2(t-t_c)} + C_0 + O(t-t_c), \quad m_{j+1} = \frac{1}{2(t-t_c)} + \tilde{C}_0 + O(t-t_c)$$

around  $t_c$ . Hence,

$$\begin{aligned} & \lim_{t \rightarrow t_c^-} \langle m_j(t)\delta(x-x_j(t)) + m_{j+i}(t)\delta(x-x_{j+1}(t)), \varphi(x) \rangle \\ &= (C_0 + \tilde{C}_0)\varphi(x(t_c)) - \lim_{t \rightarrow t_c^-} \frac{\varphi(x_j(t)) - \varphi(x_{j+1}(t))}{2(t-t_c)} \\ &= \left( \lim_{t \rightarrow t_c^-} (m_j + m_{j+1}) \right) \varphi(x(t_c)) - \frac{1}{2} \left( \lim_{t \rightarrow t_c^-} (\dot{x}_j - \dot{x}_{j+1}) \right) \varphi'(x(t_c)) \\ &= \left( \lim_{t \rightarrow t_c^-} (m_j + m_{j+1}) \right) \varphi(x(t_c)) - \frac{1}{2} \left( \lim_{t \rightarrow t_c^-} (u(x_j(t), t) - u(x_{j+1}(t), t)) \right) \varphi'(x(t_c)), \end{aligned}$$

where in the last step we have used equation (1.7) for  $b = 3$ . The claim follows now easily from the definitions of distributions  $\delta$  and  $\delta'$ .  $\square$

**Remark 3.18.** Shockpeakon creation described by Theorem 3.17 confirms the scenario that at the collision the colliding peakon-antipeakon pair creates the shock (the  $\delta'$  contribution above) and the peakon or antipeakon contribution (the  $\delta$  contribution) thus giving the overall collision data of two peakons/antipeakons and a shock. This has been previously verified for the case  $n = 2$  in [11].

## 4 Three multipeakons; spectral properties

This section addresses basic questions related to the spectral characterization of the peakon dynamics (3.2).

**Lemma 4.1.** *Let  $N^+$  denote the number of positive masses and  $n^+$  be the number of eigenvalues of the spectral problem  $A(z) = 1 - M_1 z + M_2 z^2 - M_3 z^3 = 0$  which have strictly positive real parts. Then*

$$N^+ = n^+.$$

*Proof.* The statement holds true if  $N^+ = 3$  by results in [23]; in that case the spectrum is positive and simple. Since  $m_i \mapsto -m_i, \lambda_i \mapsto -\lambda_i$  is a symmetry of the eigenvalue problem it suffices to analyze only the case with two positive masses, that is  $N^+ = 2$ .

Then

$$M_3 = \frac{1}{\lambda_1 \lambda_2 \lambda_3} < 0.$$

To prove the claim we have to exclude that three eigenvalues have strictly negative real parts (recalling that complex roots must occur in conjugate pairs) or that there is one negative and two purely imaginary conjugate roots. In either case  $M_1 = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} < 0$ , and  $M_2 = \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_3\lambda_1} > 0$ . So we assume that  $M_1 < 0$  and  $M_2 > 0$  in order to derive a contradiction.

Case 1. If  $m_1 < 0$ , then  $m_2, m_3 > 0$  and  $0 < m_2 + m_3 < -m_1$ . Therefore

$$-m_1 m_3 (1 - e^{x_1 - x_3})^2 > m_2 m_3 (1 - e^{x_2 - x_3})^2,$$

which implies

$$M_2 = m_1 m_2 (1 - e^{x_1 - x_2})^2 + m_2 m_3 (1 - e^{x_2 - x_3})^2 + m_1 m_3 (1 - e^{x_1 - x_3})^2 < 0$$

and thus leads to a contradiction.

Case 2. If  $m_3 < 0$ , it is similar to Case 1.

Case 3. If  $m_2 < 0$ , then  $m_1, m_3 > 0$  and  $0 < m_1 + m_3 < -m_2$ .

Denote  $\hat{m} = m_1 + m_3$ , then

$$m_1 m_2 (1 - e^{x_1 - x_2})^2 + m_2 m_3 (1 - e^{x_2 - x_3})^2 < -\hat{m} [m_1 (1 - e^{x_1 - x_2})^2 + m_3 (1 - e^{x_2 - x_3})^2].$$

Set  $\alpha = e^{x_1 - x_2}$ ,  $\beta = e^{x_2 - x_3}$ ,  $m_1 = \theta \hat{m}$ ,  $m_3 = (1 - \theta) \hat{m}$ , and

$$f(\theta) = \theta(1 - \alpha)^2 + (1 - \theta)(1 - \beta)^2 - \theta(1 - \theta)(1 - \alpha\beta)^2.$$

$f(\theta)$  is a quadratic function with respect to  $\theta$  with the discriminant

$$\begin{aligned} \Delta &= ((1 - \alpha)^2 - (1 - \beta)^2 - (1 - \alpha\beta)^2)^2 - 4(1 - \alpha\beta)^2(1 - \beta)^2 \\ &= -(1 - \alpha)^2(1 - \beta)^2(1 + \alpha)(1 + \beta)(3 - \alpha - \beta - \alpha\beta) < 0, \end{aligned}$$

which leads to  $f(\theta) > 0$ . Therefore

$$\begin{aligned} M_2 &< -\hat{m} [m_1 (1 - e^{x_1 - x_2})^2 + m_3 (1 - e^{x_2 - x_3})^2] + m_3 m_1 (1 - e^{x_1 - x_3})^2 \\ &= -\hat{m}^2 (\theta(1 - \alpha)^2 + (1 - \theta)(1 - \beta)^2 - \theta(1 - \theta)(1 - \alpha\beta)^2) = -\hat{m}^2 f(\theta) < 0, \end{aligned}$$

hence a contradiction.  $\square$

Clearly, by reflection symmetry, we obtain

**Corollary 4.2.** *Let  $N^-$  denote the number of negative masses and  $n^-$  be the number of eigenvalues of the spectral problem  $A(z) = 1 - M_1 z + M_2 z^2 - M_3 z^3 = 0$  which have strictly negative real parts. Then*

$$N^- = n^-.$$

Another useful corollary is that there are no eigenvalues on the line  $\operatorname{Re} z = 0$ .

**Corollary 4.3.** *None of the eigenvalues of the spectral problem  $A(z) = 1 - M_1 z + M_2 z^2 - M_3 z^3 = 0$  is purely imaginary.*

**Corollary 4.4.** *The spectral problem for  $n = 3$  can never have triple roots.*

*Proof.* Suppose, without loss of generality, that the spectral problem has triple positive roots. Then all the masses are positive, i.e. the peakons case. However the eigenvalues for the peakons are simple, hence a contradiction.  $\square$

**Remark 4.5.** Figure 1 on page 19 illustrates how the eigenvalues are distributed for the mass signature  $m_1(0) > 0, m_2(0) < 0, m_3(0) > 0$ , abbreviated  $(+ - +)$ . The graph depicts  $75 \times 75$  triples of eigenvalues for different values of masses within that configuration. The actual input data is  $m_1 = 1.2 + 0.02j$ ,  $m_2 = -5 - 0.01k$ ,  $m_3 = 4$ ,  $x_1 = -0.2$ ,  $x_2 = 0$ ,  $x_3 = 0.1$ ,  $1 \leq j, k \leq 75$ . Observe that indeed the line  $\text{Re } \lambda = 0$  contains no eigenvalues.

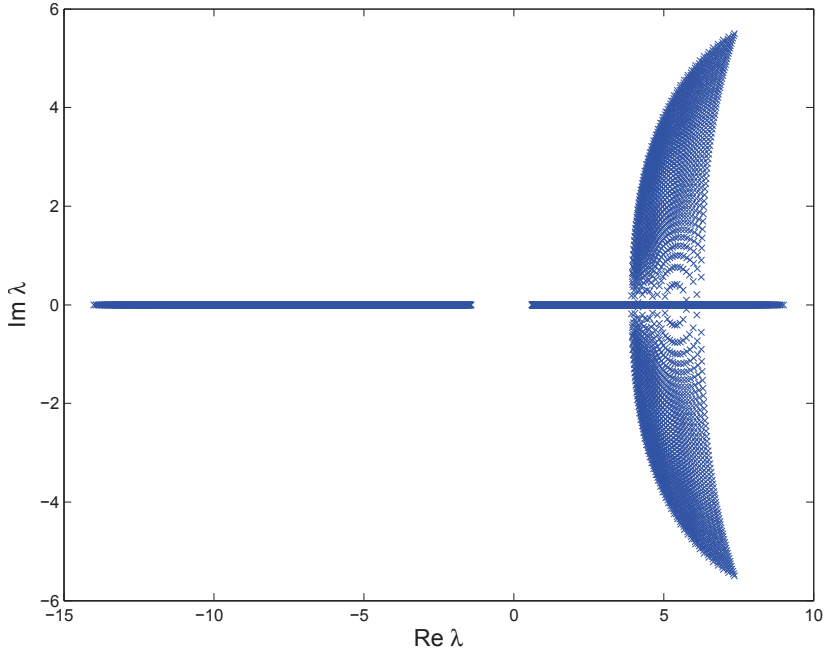


Figure 1: A portrait of eigenvalue distribution for the mass signature  $(+ - +)$

## 5 Three multipeakons; classification

The goal of this section is to understand the impact of the configuration of signs of masses on the occurrence of collisions.

One can classify the system of three multipeakons by the signs of the initial values of  $m_i$ s:

- (i)  $m_1(0) > 0, m_2(0) > 0, m_3(0) > 0$ ;    (ii)  $m_1(0) > 0, m_2(0) > 0, m_3(0) < 0$ ;
- (iii)  $m_1(0) > 0, m_2(0) < 0, m_3(0) > 0$ ;    (iv)  $m_1(0) > 0, m_2(0) < 0, m_3(0) < 0$ ;
- (v)  $m_1(0) < 0, m_2(0) > 0, m_3(0) > 0$ ;    (vi)  $m_1(0) < 0, m_2(0) > 0, m_3(0) < 0$ ;
- (vii)  $m_1(0) < 0, m_2(0) < 0, m_3(0) > 0$ ;    (viii)  $m_1(0) < 0, m_2(0) < 0, m_3(0) < 0$ .

The first and last cases are pure peakon and antipeakon, which are already well-known. In view of Lemma 3.1, the symmetry  $m_i \rightarrow -m_i, t \rightarrow -t$  reduces the eight cases to four cases. To gain some clarity we will supplement a reference to any of the cases from the list above by an ordered collection of signs, i.e. case (i) is equivalent to  $(+++)$ , case (ii) to  $(++-)$  etc. and we will refer to a given mass signature as a *mass signature*.

**Theorem 5.1.** *If the mass signature is:*

- (1)  $(--+)$  or  $(-++)$ , then no collisions will occur for positive times,
- (2)  $(+-+)$  or  $(+--)$ , then no collision will occur for negative times.

Furthermore, if the eigenvalues are not in anti-resonance, a collision will always happen at some finite time  $t_c$

- (1) if the mass signature is  $(--+)$  or  $(-++)$ , then the collision will happen at a negative time,
- (2) if the mass signature is  $(+-+)$  or  $(+--)$ , then the collision will happen at a positive time.
- (3) if the mass signature is  $(-+-)$  or  $(+ - +)$ , then the collision will happen at both a finite positive and a finite negative time.

*Proof.* First, we show that in the case of item (1) no collisions occur in positive time. Indeed, by examining the formulas (3.6) and (3.8) we see that the respective right hand sides can not be equal 0 for  $t \geq 0$ . The same argument works for item (2) and negative times.

Let  $\lambda_1, \lambda_2, \lambda_3$  be the eigenvalues of the system which we can order as  $\text{Re } \frac{1}{\lambda_1} \leq \text{Re } \frac{1}{\lambda_2} \leq \text{Re } \frac{1}{\lambda_3}$ . Since the masses have different signs, by Lemma 4.1, we have  $\text{Re } \frac{1}{\lambda_1} < 0 < \text{Re } \frac{1}{\lambda_3}$ .

*Case 1:* The eigenvalues are simple. Since the eigenvalues are not in anti-resonance, all the residues  $b_i$ 's and  $\tilde{b}_i$ 's are nonzero according to (2.28). Hence, according to (2.13) and (2.21), there exists at least one increasing and one decreasing exponential function in the expansions of  $e^{x_3}$  (respectively in the

expansions of  $e^{-x_1}$ ). Moreover, since we are squaring  $e^{x_3}$ ,  $e^{-x_1}$  respectively, the coefficient of the leading exponential will be strictly positive if the spectrum is real, or strictly positive except for a set of measure zero if the spectrum is degenerate or complex. That is to say, both integrals

$$\int_0^t e^{2x_3(\tau)} d\tau \text{ and } \int_0^t e^{-2x_1(\tau)} d\tau$$

will diverge to  $\pm\infty$  as  $t \rightarrow \pm\infty$ . Hence, there exists a positive (respectively negative) time  $t_c$  such that  $\frac{1}{m_3} = 0$  or  $\frac{1}{m_1} = 0$  whenever  $m_3(0) < 0$  or  $m_1(0) > 0$  (respectively  $m_3(0) > 0$  or  $m_1(0) < 0$ ). This proves the claim in view of lemma 3.11.

*Case 2:* There is a double root, i.e.  $\lambda_1 = \lambda_2 \neq \lambda_3$ , then  $\lambda_1$  will not be the double root of  $B(z)$  (respectively  $\tilde{B}(z)$ ), therefore at least one of  $b_1^{(1)}(0)$  and  $b_1^{(2)}(0)$  (respectively  $\tilde{b}_1^{(1)}(0)$  and  $\tilde{b}_1^{(2)}(0)$ ) is nonzero. This also implies that there exists at least one increasing and one decreasing exponential function in the expansions of both  $e^{x_3}$  and  $e^{-x_1}$ . To show that  $\lambda_1$  will not be the double root of  $B(z)$ , we only need to note that  $\lambda_1$  must be the double root of  $A(-z)C(z)$  if  $\lambda_1$  is the double root for both  $A(z)$  and  $B(z)$  according to Lemma 2.14. However,  $C(\lambda_1)$  must be nonzero, otherwise  $\Psi(z)$  is identically equal to 0 which leads to a contradiction. Therefore  $-\lambda_1$  must be the double root of  $A(z)$ , which also leads to a contradiction.  $\square$

It is immediate from the above theorem that

**Corollary 5.2.**

- (1) For cases (v)(- + +) and (vii)(- - +) there exists a unique solution to the ODEs (3.2) for all positive  $t$ .
- (2) For cases (ii)(+ + -) and (iv)(+ - -) there exists a unique solution to the ODEs (3.2) for all negative  $t$ .

The global existence (in one time direction) has an interesting impact on the spectrum of the boundary value problem.

**Theorem 5.3.** *The eigenvalues of the spectral problem of cases (ii, iv, v, vii) are real, simple, nonzero, and are equal to the inverses of asymptotic values of masses.*

*Proof.* We give a complete proof for the case (ii)(+ + -). First, we note that

$$m_3(t) = m_3(0) \exp \left( -2 \int_t^0 [m_1(s)e^{x_1(s)-x_3(s)} + m_2(s)e^{x_2(s)-x_3(s)}] ds \right) \neq 0.$$

For negative  $t$ , recalling that  $m_1, m_2$  will remain positive, we obtain

$$|m_3(t)| \leq |m_3(0)|.$$

Since  $m_1 \leq m_1 + m_2 \leq M_1 + |m_3(t)|$  we derive an upper bound on  $m_1$ , and thus on  $m_2$ , namely

$$m_i(t) \leq M_1 + |m_3(0)|, \quad i = 1, 2.$$

We observe that  $0 < M_1 + |m_3(0)|$ , otherwise  $m_1 = m_2 = 0$  for all times.

**Claim I**

$$\frac{|M_3|}{(M_1 + |m_3(0)|)|m_3(0)|} < m_i(t) < M_1 + |m_3(0)|, \quad t < 0.$$

*Proof.* (Claim I) We only need to prove the lower bound. To this end we estimate:

$$|M_3| < m_1(t)m_2(t)|m_3(t)| < m_1(t)m_2(t)|m_3(0)|,$$

and use the upper bound above on one of the factors  $m_1$  or  $m_2$ .  $\square$

Replacing in the estimate for  $|M_3|$  both factors  $m_1$  and  $m_2$  with their upper bounds we extend the claim to the bound on  $m_3$ .

**Claim II**

$$\frac{|M_3|}{(M_1 + |m_3(0)|)^2} < |m_3(t)| < |m_3(0)|.$$

**Claim III**

$$x_i(t) - x_j(t) \rightarrow -\infty, \quad (1 \leq i < j \leq 3),$$

when  $t \rightarrow -\infty$ .

*Proof.* (Claim III) The following estimate holds:

$$\begin{aligned} & \frac{|M_3|}{(M_1 + |m_3(0)|)^2 |m_3(0)|} \leq \\ & \exp\left(-2 \int_{-\infty}^0 [m_1(s)e^{x_1(s)-x_3(s)} + m_2(s)e^{x_2(s)-x_3(s)}] ds\right) \end{aligned}$$

and thus

$$\int_{-\infty}^0 [m_1(s)e^{x_1(s)-x_3(s)} + m_2(s)e^{x_2(s)-x_3(s)}] ds < \infty,$$

which, in view of the boundedness of  $m_1$  and  $m_2$ , implies

$$\int_{-\infty}^0 e^{x_i(s)-x_3(s)} ds < +\infty, \quad i = 1, 2.$$

In addition, direct estimates on equations (3.2) using the bounds on  $m_1, m_2, m_3$ , show that the velocities are bounded, which means the derivative of the integrand  $e^{x_i(s)-x_3(s)}$  ( $i = 1, 2$ ) is bounded. Therefore

$$\lim_{s \rightarrow -\infty} e^{x_i(s)-x_3(s)} = 0, \quad (i = 1, 2),$$

which is equivalent to

$$\lim_{t \rightarrow -\infty} x_i(t) - x_3(t) = -\infty, \quad (i = 1, 2).$$

Now we turn to the ODE for  $m_1$

$$m_1(t) = m_1(0) \exp \left( -2 \int_t^0 [m_2(s)e^{x_1(s)-x_2(s)} + m_3(s)e^{x_1(s)-x_3(s)}] ds \right) > 0.$$

Since  $m_1$  is, for negative  $t$ , bounded from above and from below away from 0, the integral

$$\int_{-\infty}^0 m_2(s)e^{x_1(s)-x_2(s)} ds < +\infty,$$

and  $\int_{-\infty}^0 e^{x_1(s)-x_2(s)} ds < +\infty$ . Repeating *verbatim* the arguments from the previous case, we get

$$\lim_{t \rightarrow -\infty} x_1(t) - x_2(t) = -\infty.$$

□

Now, since the improper integrals appearing in the formulas for  $m_1$  and  $m_3$  exist, we can take the limit  $t \rightarrow -\infty$ . Let us denote those limits by  $m_1(-\infty)$ ,  $m_3(-\infty)$  respectively. Using  $M_1$  we conclude that  $m_2$  also has a limit, say,  $m_2(-\infty)$ . The characteristic polynomial  $A(z)$  (see (3.3)) reads:

$$\begin{aligned} A(z) &= \left(1 - \frac{z}{\lambda_1}\right) \left(1 - \frac{z}{\lambda_2}\right) \left(1 - \frac{z}{\lambda_3}\right) \\ &= (1 - m_1(-\infty)z)(1 - m_2(-\infty)z)(1 - m_3(-\infty)z). \end{aligned}$$

**Claim IV**  $\lim_{t \rightarrow -\infty} x_1 = \lim_{t \rightarrow -\infty} x_2 = -\lim_{t \rightarrow -\infty} x_3 = -\infty$ .

*Proof.* (Claim IV) We prove first the claim for  $x_1$ . The right hand side in the equation for  $\dot{x}_1$  (see (3.2)) reads  $m_1 + m_2e^{x_1-x_2} + m_3e^{x_1-x_3}$ . When  $t \rightarrow -\infty$  the second and third terms go to 0. The limit  $m_1(-\infty) > 0$  because of the lower bound on  $m_1$ . Hence there exists a constant  $\alpha > 0$  and another constant  $t^* < 0$  such that

$$0 < \alpha < \dot{x}_1, \quad \text{for all } t \leq t^*.$$

Integrating this inequality from (negative)  $t$  to  $t^*$  we obtain  $x_1(t) \leq \alpha t + C$ , where  $C$  is a constant, which proves that  $x_1 \rightarrow -\infty$  as  $t \rightarrow -\infty$ . The same argument works for  $x_2$ . Since  $m_3(-\infty) < 0$  we get an opposite estimate for  $x_3$ , namely, there exists a constant  $\beta < 0$  and another constant  $t^{**} < 0$  such that

$$\dot{x}_3 < \beta < 0, \quad \text{for all } t \leq t^{**}$$

which, upon integration, yields  $\beta t + D \leq x_3(t)$ , forcing  $x_3(t) \rightarrow +\infty$  when  $t \rightarrow -\infty$ . □

Since  $m_3(-\infty) < 0$ , to prove simplicity, we need to show that the remaining two positive limits  $m_1(-\infty)$  and  $m_2(-\infty)$  are distinct. We adapt the proof of a similar statement in [23]. First, we observe that in view of (3.2)  $\dot{m}_2 > 0$ , hence  $m_2(t)$  is increasing. The right hand side of the equation for  $\dot{m}_1$  reads (after dividing by  $2m_1$ )  $-m_2e^{x_1-x_2} - m_3e^{x_1-x_3} = e^{x_1-x_2}(-m_2 - m_3e^{x_2-x_3})$ . Since the second term goes to 0 as  $t \rightarrow -\infty$ , and  $m_1$  is bounded away from 0, we obtain that there exists  $t^* < 0$  such that:

$$\dot{m}_1 < 0, \quad \text{for all } t < t^*,$$

So  $m_1$  is decreasing for  $t$  sufficiently large and negative. Suppose, to derive a contradiction, that  $m_1(-\infty) = m_2(-\infty)$  then

$$m_1(t) - m_2(t) < 0, \quad \text{for all } t < t^*.$$

On the other hand,

$$\int_t^0 (\dot{x}_1 - \dot{x}_2) d\tau = x_1(0) - x_2(0) - (x_1(t) - x_2(t)) = \int_t^0 (m_1 - m_2) d\tau + B(t),$$

where  $B(t)$  has a finite limit as  $t \rightarrow -\infty$ . The left hand side diverges to  $+\infty$  as  $t \rightarrow -\infty$ , while the right hand side can only diverge to  $-\infty$  based on the inequality above. This contradiction shows that  $m_1(-\infty) > m_2(-\infty)$ .

The proof for the other three cases is analogous.  $\square$

**Remark 5.4.** It is helpful to have an intuitive understanding of the above theorems. The emerging picture is this: if one has a swarm of peakons colliding with a swarm of antipeakons then the system essentially behaves as if it were a peakon-antipeakon pair. Thus the system has the following characteristics:

1. peakons and antipeakons are asymptotically (in an appropriate time direction) free with asymptotic velocities  $\dot{x}_j = \frac{1}{\lambda_j}$  which are distinct
2. peakons, antipeakons separate, that is  $x_i - x_j \rightarrow -\infty$  for  $i < j$ .

There are only two cases left:  $(-+-)$  and  $(+--)$ . In view of the reflection symmetry (see Lemma 3.1), it suffices to analyze only one of them. We choose  $(-+-)$ .

**Theorem 5.5.** *Suppose the mass signature is  $(-+-)$ , then there exists a positive time  $t_c$  such that  $x_1(t_c) < x_2(t_c) = x_3(t_c)$  and a negative time  $t_c^*$  such that  $x_1(t_c^*) = x_2(t_c^*) < x_3(t_c^*)$ .*

*Proof.* We only need to prove that  $m_1, m_2$  will never collide at a positive time. If not, assume that there exists a positive  $t_c$  such that  $m_1 \rightarrow -\infty, m_2 \rightarrow +\infty$  and  $m_3$  remains bounded when  $t \rightarrow t_c$ . Then by equation (3.2)

$$\frac{d}{dt}(x_1 - x_2) = (1 - e^{x_1-x_2})(m_1 - m_2 - m_3e^{x_2-x_3}) < 0,$$



when  $t$  is sufficiently close to  $t_c$ . This contradicts  $x_1 - x_2 \rightarrow 0$ . Likewise, for negative times, we need to eliminate a collision of  $x_2$  and  $x_3$ . However,

$$\frac{d}{dt}(x_2 - x_3) = (1 - e^{x_2 - x_3})(m_2 - m_3) - m_1(e^{x_1 - x_3} - e^{x_1 - x_2}) > 0$$

for sufficiently close to the collision time  $t_c^* < 0$ , again contradicting that  $x_2 - x_3 \rightarrow 0$  as  $t \rightarrow t_c^*+$ .  $\square$

Below we put the results of our investigation in a form of Table 1. Speaking of the asymptotic behaviour we denote by AF the system which is asymptotically free in both time directions, if it is only in one, say in the direction of positive time, then we abbreviate it as AF+, etc.

Table 1: Correlations of masses, spectrum and asymptotics

Mass signature	Spectrum	Asymptotic behaviour	Collisions
+++	+++	AF	none
++-	++-	AF-	$0 < t_c$
+ - +	$\lambda_1 < 0 < \text{Re } \lambda_2 \leq \text{Re } \lambda_3$	confined	$t_c^* < 0 < t_c$
+ --	--+	AF-	$0 < t_c$
- ++	- ++	AF+	$t_c^* < 0$
- + -	$\text{Re } \lambda_1 \leq \text{Re } \lambda_2 < 0 < \lambda_3$	confined	$t_c^* < 0 < t_c$
- - +	--+	AF+	$t_c^* < 0$
- - -	- - -	AF	none

We would like to conclude this section by discussing briefly the question of spectral data and in which sense the formulas obtained in [23] can be used to produce multipeakon solutions. We recall, in the notation of that paper,

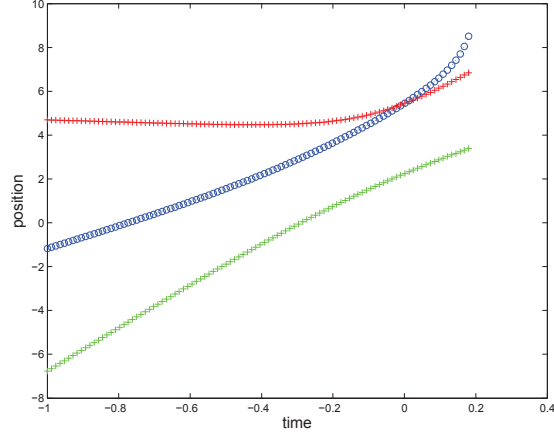
$$x_{k'} = \log \frac{U_k}{V_{k-1}}, \quad m_{k'} = \frac{(U_k)^2 (V_{k-1})^2}{W_k W_{k-1}} \quad (k = 1, \dots, n), \quad (5.1)$$

where  $k' = n + 1 - k$ , and  $U_k$  and  $V_k$  are certain rational functions of spectral data

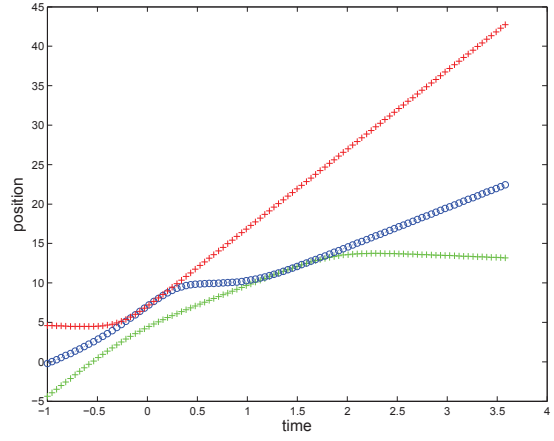
$$\mathcal{R} = \{(\lambda, b) \in \mathbf{R}^{2n} : 0 < \lambda_1 < \dots < \lambda_n, \text{ all } b_i > 0\}. \quad (5.2)$$

(see formulas (2.44) and (2.45) in [23] for definitions), while  $W_j = U_j V_j - U_{j+1} V_{j-1}$ . Clearly, for multipeakons,

1. the spectrum is no longer positive, simple, or even real,
2. the residues  $b_j$  can be negative and, in general, complex,
3. the anti-resonance condition  $\lambda_i + \lambda_j = 0$  renders the formulas not directly applicable
4. once cannot extend the formulas beyond a collision point because there is no longer guarantee that  $x_i < x_j$  for  $i < j$ .



(a) DP mass signature  $(++-)$ ;  $m_2$  collides with  $m_3$  at  $t_c = 0$ .  
Crossing of  $x_2$  with  $x_3$



(b) CH mass signature  $(++-)$ ;  $m_2$  collides with  $m_3$  at  $t_c = 0$ .  
Collision of  $m_2$  with  $m_3$ , followed by another collision of  $m_2$  with  $m_1$ . No crossing.

Figure 2: Comparison of DP and CH collisions

**Example 5.6.** Figure 2(a) on page 26 illustrates how the formulas would work for the mass signature  $(+ + -)$ . A point from the spectral set is chosen so that the collision occurs at  $t_c = 0$ . The continuation beyond the collision point would force new ordering  $x_1 < x_3 < x_2$  which means that this is not the original multipeakon problem given by equation (1.5) for  $b = 3, n = 3$ , even though the solution still satisfies equation (3.2), albeit in the wrong region. This should be contrasted with the behaviour of peakons at collision points for the CH equation as illustrated by figure 2(b) on page 26. The second particle bounces between  $m_3$  and  $m_1$  and, consequently, no change of ordering is required.

**Example 5.7.** In this example we consider the mass signature  $(+ - +)$ . Figure 3 on page 27 illustrates how the formulas (5.1) would work for the mass signature  $(+ - +)$ . In particular, as predicted by Theorem 5.5 (using the reflection symmetry), there are two collision points, one for negative time, one for positive.

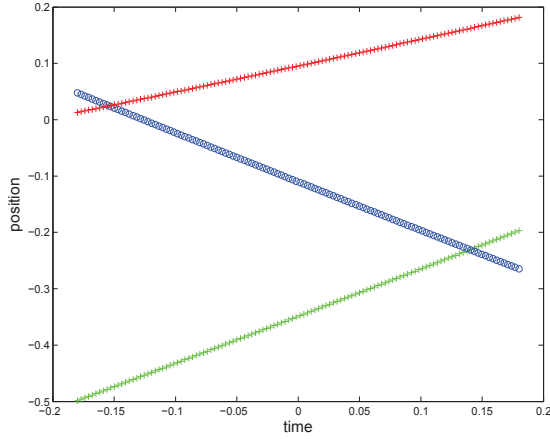


Figure 3: DP mass signature  $(+ - +)$ ;  $m_2$  collides with  $m_3$  at  $t_c^* < 0$  while  $m_2$  collides with  $m_1$  for  $t_c > 0$ . Confined state.

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